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Existence of Periodic Solutions for the Discrete-Time Counterpart of a Neutral-Type Cellular Neural Network with Time-Varying Delays and Impulses

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Abstract. From the mathematical point of view, a cellular neural network (CNN) can be characterized by an array of identical nonlinear dynamical systems called cells (neurons) that are locally interconnected. Using the semi-discretization method, in the present talk we construct a discrete-time counterpart of a neutral-type CNN with time-varying delays and impulses. Sufficient conditions for the existence of periodic solutions of the discrete-time system thus obtained are found by using the continuation theorem of coincidence degree theory.

Keywords: cellular neural networks, impulses, neutral type CNNs, time-varying delay.
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INTRODUCTION

Over the past two decades neural networks have been widely studied since they have been successfully applied to various processing problems such as optimization, image processing, associative memory and many other fields (see [4] and references given therein). Different types of applications depend on the dynamical behaviours of the neural networks. Cellular neural networks (CNNs) were introduced in the 1980s by Chua and Yang [2]. Since then, many researchers have done extensive and interesting works on this subject because of its potential applications in real life problems such as signal processing, pattern recognition, chemical processes, nuclear reactors, biological systems, static image processing, associative memories, optimization problems and so on [2, 4, 6]. A CNN is a massive parallel computing paradigm defined in a discrete N-dimensional space. The basic circuit unit of CNNs is called a cell (neuron). It contains linear and nonlinear circuit elements, which typically are linear capacitors, linear resistors, linear and nonlinear controlled sources, and independent sources. The structure of CNNs is similar to that found in cellular automata; namely any cell in a cellular neural network is connected only to its neighbor cells and the adjacent cells can interact directly with each other [2].

MAIN RESULT

We consider the following impulsive neutral-type cellular neural network with time-varying delays:

$$\dot{x}_i(t) - d_i\dot{x}_i(t-\sigma_i(t)) = -a_i(t)x_i(t) + \sum_{j=1}^{m} b_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{m} c_{ij}(t)g_j(x_j(t-\tau_{ij}(t))) + I_i(t),\ t>0,\ t\neq t_k, \tag{1}$$

$$\Delta x_i(t_k) = -\alpha_{ik}x_i(t_k) + \sum_{j=1}^{m} \beta_{ijk}\Phi_j(x_j(t_k)) + \sum_{j=1}^{m} \gamma_{ijk}\Gamma_j(x_j(t_k-\tau_{ij}(t_k))) + \zeta_{ik}, \qquad k\in\mathbb{N}, \tag{2}$$

$$x_i(s) = \varphi_i(s), \quad s\in[-\chi,0], \qquad i=\overline{1,m}, \tag{3}$$

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where $x_i(t)$ is the state of the i-th neuron at time t and $f_j(\cdot), g_j(\cdot)$ denote activation functions; the constants b_{ij}, c_{ij} represent the weights (or strengths) of the synaptic connections between the j-th neuron and the i-th neuron, respectively without and with transmission delay $\tau_{ij}(t)$; $\sigma_i(t)$ is the time delay in the state velocity $\dot{x}_i(t)$; $I_i(t)$ denotes the external bias on the i-th unit at time t; $a_i(t)$ is the rate with which the i-th unit resets its potential to the equilibrium state when isolated from the network and external inputs; t_k ($k \in \mathbb{N}$) are the moments (instants) of impulse effect satisfying $0 < t_1 < t_2 < \cdots < t_k < \cdots$ and $\lim_{k\to\infty} t_k = \infty$; $\Delta x_i(t_k) := x_i(t_k+0) - x_i(t_k-0) \equiv x_i(t_k+0) - x_i(t_k)$ represents the instantaneous change of the state of the i-th neuron at time t_k; d_i, α_{ik}, β_{ijk}, γ_{ijk}, ζ_{ik} are some constants, and $\chi > 0$ will be specified later.

Now we make the following assumptions:

[H1] (periodicity) There exists a positive number ω and a positive integer p such that

$$\begin{gathered} a_i(t+\omega) = a_i(t), \quad I_i(t+\omega) = I_i(t), \quad \sigma_i(t+\omega) = \sigma_i(t) \quad \text{for} \quad t \geq 0 \quad \text{and} \quad i = \overline{1,m}, \\ b_{ij}(t+\omega) = b_{ij}(t), \quad c_{ij}(t+\omega) = c_{ij}(t), \quad \tau_{ij}(t+\omega) = \tau_{ij}(t) \quad \text{for} \quad t \geq 0 \quad \text{and} \quad i,j = \overline{1,m}, \\ t_{k+p} = t_k + \omega \quad \text{for} \quad k \in \mathbb{N}, \\ \alpha_{i,k+p} = \alpha_{ik}, \quad \zeta_{i,k+p} = \zeta_{ik} \quad \text{for} \quad k \in \mathbb{N} \quad \text{and} \quad i = \overline{1,m}, \\ \beta_{ij,k+p} = \beta_{ijk}, \quad \gamma_{ij,k+p} = \gamma_{ijk} \quad \text{for} \quad k \in \mathbb{N} \quad \text{and} \quad i,j = \overline{1,m}. \end{gathered}$$

[H2] (boundedness, continuity and smoothness) $|d_i| < 1$ for $i = \overline{i,m}$, $d_i = 0$ whenever $\sigma_i(t) \equiv 0$; the functions $a_i(t), b_{ij}(t), c_{ij}(t)$ are continuous on $[0,\infty]$;

$$a_i(t) > 0 \quad \text{for} \quad t \geq 0 \quad \text{and} \quad \alpha_{ik} > 0 \quad \text{for} \quad k \in \mathbb{N}, \quad i = \overline{1,m};$$

there exist positive constants F_j, $G_j \tilde{F}_j \tilde{G}_j$ ($j = \overline{1,m}$) such that

$$\begin{aligned} |f_j(x) - f_j(y)| &\leq F_j|x-y|, & |g_j(x) - g_j(y)| &\leq G_j|x-y|, \\ |\Phi_j(x) - \Phi_j(y)| &\leq \tilde{F}_j|x-y|, & |\Gamma_j(x) - \Gamma_j(y)| &\leq \tilde{G}_j|x-y| \quad \text{for any} \quad x,y \in \mathbb{R}; \end{aligned}$$

the functions $\sigma_i(t)$ ($i = \overline{1,m}$) and $\tau_{ij}(t)$ ($i,j = \overline{1,m}$) are nonnegative, continuously differentiable for $t > 0$ and such that

$$\sup_{t>0} \dot{\sigma}_i(t) < 1, \quad \sup_{t>0} \dot{\tau}_{ij}(t) < 1 \qquad \text{for} \quad i,j = \overline{1,m};$$

for each $i \in \{1,2,\ldots,m\}$, either $\sigma_i(t) \equiv 0$ or $\sigma_i(t) > 0$ for $t \in [0,\omega]$; the functions $\varphi_i(s)$ ($i = \overline{1,m}$) are continuous on the interval $[-\chi, 0]$, where $\chi = \max\{\sigma, \tau\}$ and $\sigma = \max_{i=\overline{1,m}} \sup_{t>0} \sigma_i(t)$, $\tau = \max_{i,j=\overline{1,m}} \sup_{t>0} \tau_{ij}(t)$.

The existence of periodic solutions for a system similar to (1), (3) (without impulses) under assumptions contained in **H1, H2** was studied in [5]. To find an ω-periodic solution of system (1), (2) means to determine the initial functions $\phi_i(s)$ so that the solution of the initial value problem (1)–(3) is ω-periodic.

Similarly to our previous paper [1], henceforth we shall derive a discrete counterpart of system (1)–(3) using the semi-discretization method and obtain sufficient conditions for the existence of periodic solutions of the latter. In this process the differentiability of the time-varying delays will not be used.

For the sake of definiteness we assume that $\chi \leq \omega$. For a positive integer N we choose the discretization step $h = \omega/N$. For the moment we assume N so large that

$$h < \min_{k=\overline{1,p}} (t_{k+1} - t_k).$$

Then each interval $[nh, (n+1)h]$ contains at most one instant of impulse effect t_k. We also assume that $h < \min\left\{ \inf_{t>0} \sigma_i(t) \,|\, \sigma_i(t) \not\equiv 0 \right\}$.

For convenience we denote $n = [t/h]$, the greatest integer in t/h, and $n_k = [t_k/h]$. Clearly, we will have $n_{k+p} = n_k + N$ for all $k \in \mathbb{N}$. We also denote $\bar{\sigma}_i(\cdot) = [\sigma_i(\cdot)/h]$, $\bar{\tau}_{ij}(\cdot) = [\tau_{ij}(\cdot)/h]$, $N_0 = [\chi/h]$.

Let $n \in \{0\} \cup \mathbb{N}$, $n \neq n_k$. This means that the interval $[nh, (n+1)h)$ contains no instant of impulse effect t_k.

We approximate the differential equations (1) on the interval $[nh,(n+1)h]$ by

$$\begin{aligned}\dot{x}_i(t) &+ a_i(nh)x_i(t) - d_i\left[\dot{x}_i(t-\bar{\sigma}_i(nh)h) + a_i(nh)x_i(t-\bar{\sigma}_i(nh)h)\right] = -a_i(nh)d_ix_i((n-\bar{\sigma}_i(nh))h)\\ &+ I_i(nh) + \sum_{j=1}^{m} b_{ij}(nh)f_j(x_j(nh)) + \sum_{j=1}^{m} c_{ij}(nh)g_j(x_j((n-\bar{\tau}_{ij}(nh))h)), \qquad i=\overline{1,m}.\end{aligned}$$

We multiply both sides of this equation by $\exp(a_i(nh)t)$, integrate over the interval $[nh,(n+1)h]$, and then multiply by $\exp(-a_i(nh)(n+1))$. Thus we obtain

$$\begin{aligned}x_i((n+1)h) - x_i(nh) &= d_i[x_i((n+1-\bar{\sigma}_i(nh))h) - x_i((n-\bar{\sigma}_i(nh))h)] - \left(1-e^{-a_i(nh)h}\right)x_i(nh) \\ &+ \frac{1-e^{-a_i(nh)h}}{a_i(nh)}\left\{I_i(nh) + \sum_{j=1}^{m} b_{ij}(nh)f_j(x_j(nh)) + \sum_{j=1}^{m} c_{ij}(nh)g_j(x_j((n-\bar{\tau}_{ij}(nh))h))\right\}.\end{aligned}\tag{4}$$

Henceforth by abuse of notation we write $x_i(n) := x_i(nh)$ and define $\Delta x_i(n) = x_i(n+1) - x_i(n)$ ($i=\overline{1,m}$, $n \in \{0\}\cup\mathbb{N}$). For convenience we adopt the notations:

$$\begin{aligned}A_i(n) &= 1-e^{-a_i(nh)h} \qquad (i=\overline{1,m},\ n\in\{0\}\cup\mathbb{N}\setminus\{n_k\}_{k\in\mathbb{N}}),\\ I_i(n) &:= \frac{1-e^{-a_i(nh)h}}{a_i(nh)}I_i(nh) \qquad (i=\overline{1,m},\ n\in\{0\}\cup\mathbb{N}\setminus\{n_k\}_{k\in\mathbb{N}}),\\ b_{ij}(n) &:= \frac{1-e^{-a_i(nh)h}}{a_i(nh)}b_{ij}(nh) \qquad (i,j=\overline{1,m},\ n\in\{0\}\cup\mathbb{N}\setminus\{n_k\}_{k\in\mathbb{N}}),\\ \sigma_i(n) &:= \bar{\sigma}_i(nh) \qquad (i=\overline{1,m},\ n\in\mathbb{N}\setminus\{n_k\}_{k\in\mathbb{N}}),\\ \tau_{ij}(n) &:= \bar{\tau}_{ij}(nh) \qquad (i=\overline{1,m},\ n\in\mathbb{N}).\end{aligned}$$

Clearly, we have $0 < A_i(n) < 1$.

With the above notation equation (4) takes the form

$$\begin{aligned}\Delta x_i(n) &= d_i[x_i(n+1-\sigma_i(n)) - x_i(n-\sigma_i(n))] - A_i(n)x_i(n) + I_i(n)\\ &+ \sum_{j=1}^{m} b_{ij}(n)f_j(x_j(n)) + \sum_{j=1}^{m} c_{ij}(n)g_j(x_j(n-\tau_{ij}(n))), \qquad i=\overline{1,m}, \quad n\neq n_k.\end{aligned}\tag{5}$$

Next, for $n=n_k$ the interval $[nh,(n+1)h]$ contains the instant of impulse effect t_k. On this interval we approximate the impulse condition (2) by

$$\Delta x_i(n_k) = -\alpha_{ik}x_i(n_k) + \zeta_{ik} + \sum_{j=1}^{m}\beta_{ijk}\Phi_j(x_j(n_k)) + \sum_{j=1}^{m}\gamma_{ijk}\Gamma_j(x_j(n_k-\tau_{ij}(n_k))), \qquad i=\overline{1,m}, \quad k\in\mathbb{N}.\tag{6}$$

For uniformity of notation we define

$$A_i(n_k) = \alpha_{ik}, \quad I_i(n_k) = \zeta_{ik} \qquad (i=\overline{1,m},\ k\in\mathbb{N}).$$

Now the difference system (5), (6) can be written in operator form as

$$\Delta x = Hx,\tag{7}$$

where

$$\begin{aligned}(Hx)_i(n) &= -A_i(n)x_i(n) + I_i(n)\\ &+ \begin{cases} d_i[x_i(n+1-\sigma_i(n)) - x_i(n-\sigma_i(n))] + \sum_{j=1}^{m} b_{ij}(n)f_j(x_j(n)) + \sum_{j=1}^{m} c_{ij}(n)g_j(x_j(n-\tau_{ij}(n))), & n\neq n_k,\\ \sum_{j=1}^{m}\beta_{ijk}\Phi_j(x_j(n_k)) + \sum_{j=1}^{m}\gamma_{ijk}\Gamma_j(x_j(n_k-\tau_{ij}(n_k))), & n=n_k.\end{cases}\end{aligned}\tag{8}$$

From the periodicity assumptions **H1** it follows that

$$A_i(n+N)=A_i(n),\qquad \sigma_i(n+N)=\sigma_i(n),\qquad \tau_{ij}(n+N)=\tau_{ij}(n)\qquad \text{for}\quad i,j=\overline{1,m},\quad n\in\{0\}\cup\mathbb{N},$$

$$b_{ij}(n+N)=b_{ij}(n),\qquad c_{ij}(n+N)=c_{ij}(n),\qquad I_i(n+N)=I_i(n)\qquad \text{for}\quad i,j=\overline{1,m},\quad n\in\mathbb{N}\setminus\{n_k\}_{k\in\mathbb{N}}.$$

We can consider the system (7) for $n\in\{0\}\cup\mathbb{N}$, with initial conditions

$$x_i(\ell)=\varphi_i(\ell)\quad \text{for}\quad \ell=0,-1,\ldots,-N_0,\qquad i=\overline{1,m}, \tag{9}$$

where $\varphi(\ell)=(\varphi_1(\ell),\varphi_2(\ell),\ldots,\varphi_m(\ell))^T$, $\ell=0,-1,\ldots,-N_0$, are given initial vectors ($\ell=[s/h]$, $\varphi_i(\ell):=\varphi_i(\ell h)$). To find an N-periodic solution of system (7) means to determine the initial vectors $\varphi(\ell)$ so that the solution of the initial value problem (7), (9) is N-periodic.

We shall prove that under certain assumptions system (7) has at least one N-periodic solution using

Mawhin's continuation theorem [3] *Let $\mathbb{X},\mathbb{Y}$ be real Banach spaces, let $L:\operatorname{Dom}L\subset\mathbb{X}\longrightarrow\mathbb{Y}$ be a Fredholm mapping of index zero, let Q be a continuous projector $\mathbb{Y}\longrightarrow\mathbb{Y}$ such that $\operatorname{Ker}Q=\operatorname{Im}L=\operatorname{Im}(I-Q)$, and let J be an isomorphism $\operatorname{Im}Q\longrightarrow\operatorname{Ker}L$.*

Let $\Omega\subset\mathbb{X}$ be an open bounded set and let $H:\mathbb{X}\longrightarrow\mathbb{Y}$ be a continuous operator which is L-compact on $\overline{\Omega}$. Assume that the following conditions hold:

(a) *for each $\lambda\in(0,1)$, $x\in\partial\Omega\cap\operatorname{Dom}L$, $Lx\neq\lambda Hx$;*

(b) *for each $x\in\partial\Omega\cap\operatorname{Ker}L$, $QHx\neq 0$;*

(c) $\deg(JQH,\Omega\cap\operatorname{Ker}L,0)\neq 0$*, where* $\deg(\cdot)$ *is the Brouwer degree.*

Then the equation $Lx=Hx$ has at least one solution in $\overline{\Omega}\cap\operatorname{Dom}L$.

For the sake of brevity we have omitted some definitions.

Let us choose $\mathbb{X}=\mathbb{Y}=\{x(n)=(x_1(n),x_2(n),\ldots,x_m(n))^T : x(n+N)=x(n),\ n\in\{0\}\cup\mathbb{N}\}$ which is a Banach space equipped with the norm

$$\|x\|=\sum_{i=1}^{m}\max_{n=\overline{0,N-1}}|x_i(n)|.$$

For $x\in\mathbb{X}$, let Hx be defined by (8), $Lx=\Delta x$. Now it remains to verify the applicability of Mawhin's theorem under certain assumptions.

REFERENCES

1. H. Akça, V. Covachev, Z. Covacheva and S. Mohamad, *Funct. Differ. Equ.* **16**, 53–72 (2009).
2. L. O. Chua and Y. Yang, *IEEE Trans. Circuits Systems* **35**, 1257–1272 (1988).
3. R. E. Gaines and J. L. Mawhin, *Coincidence Degree and Nonlinear Differential Equations*, Springer-Verlag, Berlin, 1977.
4. A. I. Galushkin, *Neural Networks Theory*, Springer, 2007.
5. Y. Li, L. Zhao and X. Chen, *Appl. Math. Model.* **36**, 1173–1183 (2012).
6. I. M. Stamova and R. Ilarionov, *Comput. Math. Appl.* **59**, 3508–3515 (2010).